Lifted Inference for Relational Continuous Models

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Abstract
Relational Continuous Models (RCMs) represent joint probability densities over attributes of objects, when the attributes have continuous domains. With relational representation, they can model joint probability distributions over large numbers of variables compactly in a natural way. This paper presents the first exact inference algorithm for RCMs at a lifted level, thus it scales up to large models of real world applications. The algorithm applies to relational pairwise models, which are (relational) products of potentials of arity 2. Our algorithm is unique in two ways. First, it is an efficient lifted inference algorithm. When Gaussian potentials are used, it takes only linear time while existing methods take cubic time. Second, it is the first exact inference algorithm which handles RCMs in a lifted way. The algorithm is illustrated over an example from Econometrics. Experimental results show that our algorithm outperforms both a ground-level inference algorithm and an algorithm built with previously-known lifted methods.

1 Introduction
Many real world systems are described by continuous variables and relations among them. Such systems include measurements in environmental-sensors networks, localizations in robotics, and economic forecasting in finance. Once a relational model among variables is given, inference algorithms can solve the problems of value prediction and classification.

At a ground level, inference with a large number of continuous variables is nontrivial. Typically, inference is the task of calculating a marginal over variables of interest. Suppose that a market index has a relationship with revenues of n banks. When marginalizing the market index out, the result is a function of n variables (revenues of banks), thus following marginalizations become harder. When n grows, the computation becomes expensive. When relations among variables follow Gaussian distributions, the computational complexity of the inference problem is cubic to the number of ground variables. Thus, computation with such models is limited to moderate-size models, preventing usage of such models for large, real-world applications.
We present a new relational model for continuous variables, where the set of continuous variables is defined as a set of continuous random variables whose range are $(-\infty, \infty)$. Each relational atom $X$ is defined over two relational atoms, $[X_1(\theta_1), \cdots, X_n(\theta_n)]$ to non-negative real numbers. For example, the potential function for a relational atom $X$ is defined as $w(x) = \prod_{i=1}^{n} w_i(x_i)$.

For example, consider the model in Figure 1. $S$ and $B$ in $L$ are two objects which represent markets and banks respectively. $S$ can be substituted by a specific market sector (e.g. $S = \text{'stock'}$). A parfactor $f_1 = \{\text{[Market[S], Loss[S, B]], } \theta\}$ is defined over two relational atoms, $[\text{Market[S], Loss[S, B]}]$. $\text{Market(auto)}$ represents the quarterly market change (e.g. $\text{Market(auto)} = -3.1\%$). $\text{Loss(auto}, \text{Pacific Bank)}$ represents the loss of the bank in the auto market. Given two values, a potential $\phi_1(\text{Market(auto)}, \text{Loss(auto}, \text{Pacific Bank}))$ provides the probability density.

### 3 Algorithm Overview for RCMs

One inference task with such models is to find the conditional density of query variables given observations of some variables in the model. Our inference algorithm, First-Order Variable Elimination (FOVE)-Continuous, recursively eliminates relational atoms as described in Figure 2.

First, it splits (terminology of (Poole 2003); shattering in (de Salvo Braz, Amir, and Roth 2005)) relational atoms such that groundings, $RV(X)$ or $RV(Y)$, of every pair of relational atoms, $X, Y$, are disjoint. It introduces observations of groundings as separate relational variables. For example, observing $\text{Market(auto)} = 30\%$ creates two separate relational atoms: $\text{Market(auto)}, \text{Market(M)_{Market}}$. The ‘$M \neq auto$’ then appears in parfactors relating to the latter relational atom. After split, FIND-ELIMINABLE finds a relational atom which satisfies conditions for one of the elimination algorithms: Inversion-Elimination (Sections 5.1) and Relational-Atom-Elimination (Section 5.2). The found atom is eliminated by our ELIMINATE-CONTINUOUS algorithm explained in Sections 4 and 5. It iterates the elimination until only query variables are remained.

Our main contributions are the algorithm ELIMINATE-CONTINUOUS, a lifted variable eliminations for continuous variables. We describe it in detail in Section 4 and 5.

### 4 Inference with Gaussian Potentials

This section presents our first main technical contribution, efficient variable elimination algorithms for relational Gaus-
4.1 Relational Pairwise Potentials

This section focuses on the product of potentials which we call Relational Normals (RNs). An RN is the following function with arity 2 (Section 5 provides a generalization for arbitrary potentials):

$$\phi_{R}(X, Y) = \prod_{x \in X \land y \in Y} \phi_{RN}(x, y) = \prod_{x \in X \land y \in Y} \frac{1}{\sqrt{2\pi\sigma}} \exp \left( -\frac{(x - y)^2}{2\sigma^2} \right)$$

Figure 3: This figure shows a challenging problem in a RCM when eliminating a set of variables (Revenue[B]). Eliminating Revenue[B] in $\phi_4$ generates an integral $\phi_5$ that makes all variables in Market[S] ground. Thus, the elimination makes the RCM into a ground network.

Figure 4: This figure shows our method for the problem shown in Figure 3. When eliminating Revenue[B], we do not generate a ground network. Instead, we directly generate the pairwise form which allows the inference at the lifted level.

This potential indicates that the difference between two random variables follows a Gaussian distribution.

Consider the models shown in Figure 3 and 4. These random variables model the relationship between each market change and the revenue of each bank. To simplify notations, we respectively shorten Market(s), Loss(s,b) and Revenue(b) to $M(s), G(s,b)$ and $R(b)$ in equations. The potential $\phi_4$ in these figures is $\phi_{RN}(M(s), R(b))$, and the complete model is $\prod_{s \in S \land b \in B} \phi_{RN}(M(s), R(b))$.

Figure 4 shows that marginalizing out a random variable $R(b)$ from the joint density results in the product RNs again (c and c’ are constants).

Formally,

$$\int_{\mathbb{R}^{|S|}} \prod_{s \in S} \phi_{RN}(M(s), R(b)) = c \cdot \exp \left( \frac{\sum_{s \in S} M(s)^2}{2\sigma^2} - \frac{\sum_{s \in S} M(s)}{2\sigma} \right)$$

$$= c \cdot \prod_{s \in S} \exp \left( \frac{1}{2\sigma^2} \sum_{b \in B} (M(s,b) - M(s))^2 \right) = c' \cdot \prod_{s \in S} \phi_{RN}(M(s), M(s))$$

Definition 1 (Connected Relational Normal) The product of RNs is connected, when the connectivity graph of RNs is a connected component. Each vertex of the connectivity graph is a random variable or a constant in RNs, and each edge is a potential (RN).

Lemma 1 The product of RNs is a probability density function when it is connected, and at least a RN includes a constant argument.

Lemma 1 can be proved by that the product of connected RNs integrates to a constant given a constant argument. However, we omit the proof for lack of space.

4.2 Constant Time Relational Atom Eliminations

We provide two constant time elimination algorithms for RNs involving a single relational potential (i.e. the product of RNs over different instances of relational atoms). The algorithms eliminate variables, while maintaining the product of RNs.

Elimination of a relational atom $X$ from $\phi_{RN}(X, Y)$ The first problem is to marginalize a relational atom $(X)$ in the product of RNs with two relational atoms $(X, Y)$:

$$\int_{\mathbb{R}^{|S|}} \exp \left( -a R(b_1) + 2b R(b_1) - c \right) = \sqrt{e} \exp \left( \frac{b^2}{a} - c \right).$$

2Note, $\int_{\mathbb{R}^{|S|}} \exp \left( -a R(b_1) + 2b R(b_1) - c \right)$.
\( \phi_{RN}(X, Y) \). The potential is the product of \(|X| \cdot |Y|\) RNs. Note that each random variable in \(X\) has a relation with each variable in \(Y\).

**Algorithm ‘Pairwise Constant’**. It marginalizes \(x_i\) in \(X\), and converts the potential into a pairwise form.

\[
\int \prod_{j \neq i} \exp \left( -\frac{(x_i - y_j)^2}{2\sigma^2} \right) = \prod_{j \neq i} \exp \left( -\frac{(y_i - y_j)^2}{2\sigma^2} \right) \tag{2}
\]

Note that the marginal over \(x_i \in X\) and the marginal over \(x_j \in X\) (\(i \neq j\)) are same. Thus, the following result is derived when it marginalizes all variables in \(X\).

\[
\int \prod_{i \neq j \in [n]} \exp \left( -\frac{(x_i - y_j)^2}{2\sigma^2} \right) = \prod_{j \neq i} \exp \left( -\frac{(y_i - y_j)^2}{2\sigma^2} \right) \tag{3}
\]

The result of integration is the product of pairwise RNs \((\phi_{RN}(X, Y))\) with the parameter \(\frac{|X|}{\sqrt{|Y|}}\).

**Theorem 2** For the product of RNs between two relational atoms, ‘Pairwise Constant’ eliminates all variables in a relational atom at a constant time.

**Proof** Eliminating a variable \(x_i\) in \(X\) takes a constant time as Equation 2. Eliminating other variables in \(X\) can be done without iterations as shown in Equation 3. Thus, the computation takes only a constant time.

**Elimination of \(n\) random variables from \(\phi_{RN}(X, X)\)**

The second problem is to marginalize some \((n)\) variables in a relational atom \((X)\) in the product of RNs within the relational atom: \(\phi_{RN}(X, X)\). The potential is the product of \(\frac{|X|^{|X|}!}{m!} \) pairwise RNs between two variables in \(X\).

**Algorithm ‘Pairwise Constant2’**. It updates the marginal after eliminating a random variable without iterations. When it eliminates \(x_m\), it calculates the parameters of \(\phi_{RN}'\) given \(\phi_{RN}\) as the following equation.

\[
\int \prod_{n \neq m} \phi_{RN}(x_m, x_n) = \prod_{n \neq m} \phi_{RN}(x_n, x_m) \cdot \int \prod_{n \neq m} \prod_{i \neq j \in [m-1]} \exp \left( -\frac{(x_i - x_j)^2}{2\sigma^2} \right) \tag{4}
\]

The coefficient of \(\phi_{RN}'\) is the sum of coefficient of \(\phi_{RN}\), \(\sigma^2\), and the coefficient of \(\phi_{RN}'\), \(\sigma^2(m-1)\). The sum of two co-efficients results in \(\sigma^2 \cdot \frac{m-1}{m-2}\). Similarly, eliminating the next random variable \(x_{m-1}\) results in \(\sigma^2 \cdot \frac{m-2}{m-3}\). Thus, eliminating \(n\) random variables results in \(\sigma^2 \cdot \frac{m-n}{m-n}\) without iterations.

**Theorem 3** For the product of RNs within a relational atom, ‘Pairwise Constant2’ eliminates \(n\) variables in the relational atom at a constant time.

**Proof** Updating the parameter of \(\phi_{RN}(X, X)\) from \(\sigma^2\) to \(\sigma^2 \cdot \frac{m-n}{m}\) takes only a constant time.

### 4.3 A Linear Time Relational Atom Elimination

This section provides a linear time variable elimination algorithm \(O(|U|)\) which can be applied to any product of RNs when the constant time algorithms of the previous sections are not applicable.

**Elimination of relational atoms from \(\prod \phi_{RN}(X_i, X_i)\)**

This problem is to marginalize some variables in \(U\), \((U = \{X_1, X_2, \ldots, X_{|U|}\})\) in the product of RNs between two relational atoms: \(\prod \phi_{RN}(X_i, X_i)\). If all relational atoms are related each other, there are \(\frac{|N|(|N|-1)}{2}\) pairwise RNs.

**Lemma 4** For \(|U|\) variables in \(|N|\) relational atoms \((U = \{X_1, X_2, \ldots, X_{|U|}\})\) and RN potentials, marginalizing \(n\) variables in a ground model takes \(O(n \cdot |U|^2)\).

**Proof** Suppose we eliminate a variable \(x \in U\). Eliminating a variable \(x\) in RN needs updates coefficients of terms \((x_i, x_j)\) where \(x_i\) and \(x_j\) have relations with the variable \(x\). When \(x\) has relations with all other variables in \(U\), the number of terms is bounded by \(O(|U|^2)\). Thus, eliminating \(n\) variables takes \(O(n \cdot |U|^2)\) because it needs \(n\) iterations.

Thus, any inference algorithm in a ground model has the order of \(O(|U|^2)\) time complexity, when it eliminates all ground variables except a few query variables.

**Algorithm ‘Pairwise Linear’**. To reduce the time complexity, our lifted algorithm uses following notations which refer multiple variables in an atom: \(X_1^{[m]} = \sum_{1 \leq i \leq m} x_i; X_1^{[mp]} = \sum_{1 \leq i \leq m} x_i^2; X_1^{[m]}[m] = \sum_{1 \leq i \leq m} x_i \cdot x_j; (X_1^{[mp]})^2 = X_1^{[mp]} + 2X_1^{[mp]}[m].\) The notations give the following properties:

\[
\exp(2X_1^{[m]}[m] - mX_1^{[mp]}) = \prod_{1 \leq i \leq m} \exp(-x_i)^2 = \phi_{RN}(X, X) \tag{5}
\]

\[
\exp(2X_1^{[m]}[m] - mX_1^{[mp]} - nX_1^{[mp]} - mY_1^{[mp]}) = \prod_{1 \leq i \leq m} \exp(-x_i)^2 = \phi_{RN}(X, Y) \tag{6}
\]

For a potential over \(X, Y\), and \(\{x\}\), it marginalizes \(x\):

\[
\int_{x'} \phi_{RN}(X, X') \cdot \phi_{RN}(Y, x') = \int_{x'} \exp\left(-m(x + n)^2 + 2(X_1^{[m]} + Y_1^{[m]}) - (X_1^{[mp]} + Y_1^{[mp]}) \right) \tag{7}
\]

\[
\cdot \exp\left(2X_1^{[m]}[m] + 2X_1^{[mp]} + 2Y_1^{[mp]} - (m+n)(X_1^{[mp]} + Y_1^{[mp]}) \right) \tag{8}
\]

\[
\cdot \exp\left(-x_i^2 - y_i^2 \right) = \phi_{RN}(X, X) \cdot \phi_{RN}(X, Y) \cdot \phi_{RN}(Y, Y) \tag{9}
\]

It iterates until all \(n\) variables are eliminated.

**Theorem 5** For \(|U|\) variables in \(|N|\) relational atoms \((U = \{X_1, X_2, \ldots, X_{|U|}\})\) and potentials in RN, ‘Pairwise Linear’ eliminates \(n\) variables in \(O(n \cdot |N|^2)\).

**Proof** WLOG, we marginalize a variable \(x'\) \(\in X_1\). We make an artificial atom \(Y\) which includes all relational atoms which have relationships with \(X_1\). Then, \(\{x'\}\) is separated from \(X_1\) \((X_1' = X_1 \setminus \{x'\})\). When marginalizing \(\phi_{RN}(X_1', X_1')\), \(\phi_{RN}(Y, x')\) over \(x'\), the marginal is also the product of RNs as Equation 4: \(\phi_{RN}'(X_1', X_1') \cdot \phi_{RN}(X_1', Y) \cdot \phi_{RN}(Y, Y)\).

The marginal can be represented without the artificial atom \(Y\). We remove \(Y\) from \(\phi_{RN}'(X_1', Y)\) and \(\phi_{RN}(Y, Y)\). \(\phi_{RN}(X_1', Y)\) is represented as the product of RNs between atoms in \(Y\) and \(X_1'\): \(\prod_{X \in Y} \phi_{RN}(X_1', X)\). \(\phi_{RN}(Y, Y)\) is also
represented as the product of RNs between atoms in B:

\[ \prod_{X \in X(Y)} \phi_{\text{RN}}(X, X \prime) \]

For each elimination, it updates parameters of all pairs \( O(|\mathcal{N}|^2) \) among \(|\mathcal{N}|\) atoms. Thus, computational complexity to eliminate \( n \) variables is the order of \( O(n \cdot |\mathcal{N}|^2) \). ■

Thus, ‘Pairwise Linear’ has the order of linear \( O(|\mathcal{U}|) \) time complexity with respect the number of ground variables.

5 Exact Lifted Inference with RCM

This section presents our algorithm, ELIMINATE-CONTINUOUS, which generates a new parfactor after eliminating a set of relational atoms given a set of parfactors. A potential of each parfactor is the product of Relational Pairwise Potentials (RPPs):

\[ \phi_{\text{RN}}(X, Y) = \prod_{x \in X \cap Y} \phi_{\text{RN}}(x, y) \]

A relational pairwise model is a RCM whose potentials are RPP. Here, RPPs are not limited to the RNs in Section 4.1.

Conditions for Exact Lifted Inference

The lifted ELIMINATE CONTINUOUS algorithm provides the exact solution for potentials of parfactors when the potentials satisfy three conditions: (I) analytically integrable; (II) closed under product operation; and (III) represented as the product of relational pairwise potentials after marginalizations. RNs are examples which satisfy the conditions.

5.1 Inversion-Elimination

Inversion elimination is applicable when the set of objects in \( g \) is same with the set of objects in \( e \), \( LV(e) = LV(g) \). Let \( \theta_1, \ldots, \theta_n \) be enumeration of \( \Theta_{\theta} \).

\[
\int_{RV(g)} \phi(g) = \int_{RV(g)} \prod_{e \in \theta} \phi_e(A_e) = \prod_{e \in \theta} \int_{RV(e)} \phi_e(A_e) \]

\[
= \prod_{e \in \theta} \int_{RV(e)} \phi_e(A_e) \mid \sigma(u) = \prod_{e \in \theta} \int_{RV(e)} \phi_e(A_e) \mid \sigma(u) = \prod_{e \in \theta} \int_{RV(e)} \phi_e(A_e) = \phi'(g) \]

Return to the financial market example, inversion elimination can eliminate \( G[S, B] \). Before elimination, we get a parfactor \( g = ((S, B), T, (M[S], G[S, B], R[B]), \phi_2 \cdot \phi_3) \) which combines the two parfactors, \( ((S, B), T, (M[S], G[S, B], R[B]), \phi_2) \) and \( ((S, B), T, (G[S, B], R[B]), \phi_3) \). Then, the elimination procedure is as follows.

\[
\int_{RV(g)} \phi(g) = \int_{RV(g)} \prod_{e \in \theta} \phi_e(M(s), G(s, b), R(b)) \]

\[
= \prod_{s \in S_{\text{auto}}, b \in B_{\text{stock}}} \left( \int_{RV(g)} \phi_e(M(s), G(s, b), R(b)) \right) \]

\[
= \prod_{s \in S_{\text{auto}}, b \in B_{\text{stock}}} \phi_{\text{new}}(M(s), R(b)) = \phi_{\text{new}}(M[S], R[B]) = \phi'(g) \]

Note that, the number of substitutions \( |\Theta_{\theta}| \) is the number of market sectors \(|S|\) times the number of banks \(|B|\). Regardless the number of substitutions, we can apply the same integration to eliminate \( |\Theta_{\theta}| \) number of random variables \( G(s, b) \). Thus, it calculates the integral \( \int_{\phi_{\text{new}}(M(s), G(s, b), R(b))} \) once regardless of specific \( s \) and \( b \). The marginal becomes the potential \( \psi'_{\text{new}}(M[S], R[B]) \) of the output parfactor \( \phi'(g) \).

5.2 Relational-Atom-Elimination

Relational-Atom-Elimination marginalizes atoms when Inversion-Elimination is not applicable. It is a generalized algorithm of those for RN shown in Section 4. It marginalizes each relational atom of a parfactor \( g \) according to three cases: (1) variables in the atom \( e \) has no direct relationship each other (i.e. \( \phi(X, Y) \phi(X, Z) \)); (2) variables in the atom \( e \) has relationships only each other (i.e. \( \phi(X, Y) \)); and (3) other cases (i.e. \( \prod \phi(X, Y, Z) \)).

For the case (1), a generalized ‘Pairwise Constant’ eliminates an atom \( e \). In this case, marginalizing a random variable in the atom does not affect marginalizing another variable in the atom as shown in Section 4.2. That is, \( \int_{RV(\theta)} \phi(S) = \int_{RV(\theta)} \phi_1(A_1, \ldots, A_n) \phi_2(A_2, \ldots, A_n) \phi_3(A_3, \ldots, A_n) \)

\[
= \prod_{\theta \in \Theta} \int_{RV(\theta)} \phi(S) = \prod_{\theta \in \Theta} \phi'(Rv(\theta)) \mid \sigma(\text{condition(1)}) \]

The marginal \( \psi'(\text{RV}(\theta)) \) is not a relational pairwise potential anymore, because all random variables in \( E \) are arguments of the potential. When condition (III) is satisfied, the marginal can be converted into the product of relational pairwise potentials: \( \psi'_{\text{new}}(\text{RV}(\theta)) = \prod_{S, X \in \text{RV}(\theta)} \phi_{\text{new}}(X, X) \).

In the econometric example, it eliminates \( R[B] \) as follows.

\[
\int_{RV[R]} \phi(g) = \int_{RV[R]} \prod_{e \in \Theta_{\theta}} \phi_{\text{new}}(M(s), R(b)) \]

\[
= \prod_{s \in S_{\text{stock}}} \prod_{b \in B_{\text{stock}}} \phi_{\text{new}}(M(s), R(b)) = \prod_{s \in S_{\text{stock}}} \phi_{\text{new}}(M(\text{auto}), \ldots, M(\text{stock})) \]

\[
= \phi_{\text{new}}(M(\text{auto}), \ldots, M(\text{stock}))^{\text{RV}(\theta)} = \phi_{\text{new}}(M(\text{auto}), \ldots, M(\text{stock})) \]

Beyond Relational Gaussian defined in Section 4.1, any potential function satisfying the third condition can convert the potential \( \psi'_{\text{new}} \) into the pairwise form \( \psi'_{\text{new}} = \prod_{e \in \Theta_{\theta}} \phi_{\text{new}}(M(s), M(b)) \).

For the cases (2) and (3), generalized algorithms of ‘Pairwise Constant’ and ‘Pairwise Linear’ are also applied, respectively.

6 Experiments

We report experiments for the recession model provided in the paper. For experiments, we implemented three algorithms: (A) inference with a grounded model; (B) inference with only Inversion-Elimination; and (C) inference with both Inversion-Elimination and Relational-Atom-Elimination. Our new algorithm (C) is significantly faster than the grounded model (A) and Inversion-Elimination (B). Note that Inversion-Elimination (B) is also our new algorithm for continuous variables, even though comparable elimination methods for discrete variables (de Salvo Braz, Amir, and Roth 2005; Milch et al. 2008; Pfeffer et al. 1999) existed prior to ours. Our experimental results are shown in Figures 5 and 6.
Inference with Gaussian distributions is a traditional problem (Roweis and Ghahramani 1999). In detail, calculating conditional densities of multivariate Gaussians requires matrix inversions (Kotz, Balakrishnan, and Johnson 2000) which are intractable for high dimensions. (Paskin 2003) shows that efficient inference is possible for a linear Gaussian when the treewidth of the model is small. For models with large treewidth, however, those inference algorithms over ground models are not applicable in practice.

8 Conclusion

In this paper, we propose a new exact lifted inference algorithm for Relational Continuous Models (RCMs). This algorithm is an advancement of exact inference in RCMs, since all previous works are restricted to discrete domains. Given a query and observations, our algorithm computes the conditional density of the query efficiently.

9 Acknowledgment

This material is supported by NSF CAREER 05-46663 and UIUC/NCSA AESIS 251024 grants.

References

Milch et al. 2008; Pfeffer et al. 1999 over discrete domains.

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7 Related Works

(Poole 2003) solves inference problems with the unification which dynamically splits a set of ground nodes and unifies them. With a counting formula, (de Salvo Braz, Amir, and Roth 2005; 2006) provide tractable algorithms. (Milch et al. 2008) devises an improved algorithm using the counting formula to represent conditional density tables compactly. However, such lifted inference algorithms for discrete variables are not applicable to continuous variables.

Markov Logic Network (MLNs) (Richardson and Domingos 2006) use first-order logic sentences to represent relationships over nodes in a graphical model. In this regard, MLNs also represent graphical models at the relational level. (Singla and Domingos 2008) provides an approximated lifted inference algorithm over discrete domain. (Domingos and Singla 2007) makes an analysis for infinitely many discrete variables. However, these achievements are not for continuous domains, too. Thus, they are comparable to lifted inferences (de Salvo Braz, Amir, and Roth 2005; Milch et al. 2008; Pfeffer et al. 1999) over discrete domains.